Topic: ∘ Path Independence and Gradient Field
∘ Curl and Green's thm.

Recall: Last time, we show that if $F = \nabla f$ (Gradient Field; Potential Function), then $\forall$ smooth curve C parametrized by $\gamma : [0,1] \to \mathbb{R}^2$, we have $\int_C \vec{F} \cdot d\vec{\gamma} = \int_C \nabla f \, d\vec{\gamma} = f(\gamma(1)) - f(\gamma(0))$.

So we are interested to know when a vector field is a Gradient Field and how to find its potential function.

Path Independence and Gradient Field:

Thm: Let $D \subseteq \mathbb{R}^2$ be an open, connected and simply connected (i.e. no 'hole') region.
Let $C \subseteq D$ be a piecewise smooth curve on D.
Let F be a smooth vector field on D.
Then the followings are equivalent:
(1) F is conservative on D.
(2) $F = \nabla f$ on D
(3) $\oint_C \vec{F} \cdot d\vec{\gamma} = 0 \quad \forall$ loop C on D
(4) $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, where $F(x,y) = (M(x,y), N(x,y))$.

Rmk: Simply connectedness means that every loops on the region can be shrinked into a point.
Examples: Simply-connected: $\mathbb{R}^2$, $\mathbb{R}^2 \setminus \{\text{positive } x\text{-axis}\}$.
Non-simply-connected: $\mathbb{R}^2 \setminus \{\text{finite no. of points}\}$

Finding Potential Function:

Example: 1) $F(x,y) = (y\cos x + y^2, \sin x + 2xy - 2y)$ on $\mathbb{R}^2$.

Check: $\dfrac{\partial M}{\partial y} = \cos x + 2y = \dfrac{\partial N}{\partial x}$

By thm above, potential function exists.

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function s.t. $F = \nabla f$.

Then we have
$$\begin{cases} \dfrac{\partial f}{\partial x} = y\cos x + y^2 & \text{— (1)} \\ \dfrac{\partial f}{\partial y} = \sin x + 2xy - 2y & \text{— (2)} \end{cases}$$

$$(1) \Rightarrow \frac{\partial f}{\partial x} = y\cos x + y^2$$

$$\Rightarrow f(x,y) = y\sin x + xy^2 + C(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = \sin x + 2xy + C'(y)$$

$$(2) \Rightarrow C'(y) = -2y, \quad C(y) = -y^2 + C$$

$\therefore$ $f(x,y) = y\sin x + xy^2 - y^2$ is a potential function of F.

2) $F(x,y) = \left(\dfrac{2x}{y}, \dfrac{1-x^2}{y^2}\right)$, $D = \{(x,y) \mid y > 0\}$

Check: $\dfrac{\partial M}{\partial y} = -\dfrac{2x}{y^2} = \dfrac{\partial N}{\partial x}$

$\Rightarrow$ Potential function exists.

Let $F = \nabla f$. Then we have
$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{2x}{y} & \text{— (1)} \\ \dfrac{\partial f}{\partial y} = \dfrac{1-x^2}{y^2} & \text{— (2)} \end{cases}$$

$$① \Rightarrow \frac{\partial f}{\partial x} = \frac{2x}{y} \Rightarrow f(x,y) = \frac{x^2}{y} + C(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{-x^2}{y^2} + C'(y)$$

$$② \Rightarrow C'(y) = \frac{1}{y^2} \Rightarrow C(y) = \frac{-1}{y} + C.$$

$\therefore f(x,y) = \frac{x^2-1}{y}$ is a potential function of F.

Q: What happens if D is not simply connected?

3) $F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$, $D = \{(x,y) \mid y > 0\}$

Check: $\frac{\partial M}{\partial y} = \frac{y^2-x^2}{x^2+y^2} = \frac{\partial N}{\partial x}$

$\Rightarrow$ Potential function exists.

Let $F = \nabla f$. Then
$$\begin{cases} \frac{\partial f}{\partial x} = \frac{-y}{x^2+y^2} & — ① \\ \frac{\partial f}{\partial y} = \frac{x}{x^2+y^2} & — ② \end{cases}$$

Recall: $\int \frac{1}{1+a^2}\, da = \tan^{-1} a + C$.

$$① \Rightarrow \frac{\partial f}{\partial x} = \frac{-y}{x^2+y^2}$$

$$\Rightarrow f(x,y) = \int \frac{-y}{x^2+y^2}\, dx + C(y)$$

$$= -\int \frac{d\left(\frac{x}{y}\right)}{1+\left(\frac{x}{y}\right)^2} + C(y)$$

$$= -\tan^{-1}\left(\frac{x}{y}\right) + C(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{x}{x^2+y^2} + C'(y)$$

② $\Rightarrow C'(y) = 0 \Rightarrow C(y) = C.$

$\therefore f(x,y) = -\tan^{-1}\left(\frac{x}{y}\right)$ is a potential function.

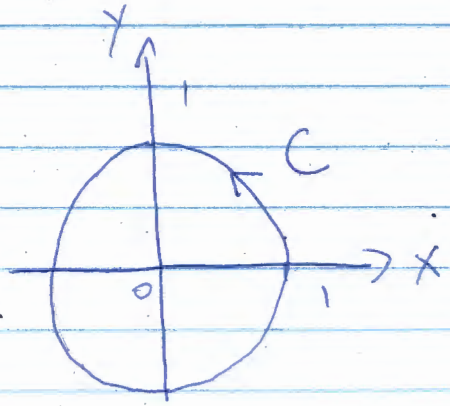
Note that $\lim_{(x,y)\to(0,0)} F(x,y)$ does not exist (exercise),

but it is well-defined elsewhere.

Hence, we can extend F to the domain $\mathbb{R}^2\setminus\{0\}$.

Note that ① $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

② On the unit circle C parametrized by
$\gamma: [0,2\pi] \to \mathbb{R}^2$, $\gamma(\theta) = (\cos\theta, \sin\theta)$,

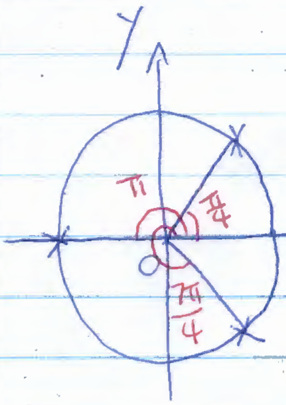


$$\oint_C \vec{F}\cdot d\vec{\gamma} = \int_0^{2\pi} (-\sin\theta, \cos\theta)\cdot(-\sin\theta, \cos\theta)\,d\theta$$
$$= \int_0^{2\pi} d\theta$$
$$= 2\pi$$
$$\neq 0.$$

Reason: 1) $\mathbb{R}^2\setminus\{0\}$ is not simply connected
So the thm fails.

2) Note that the potential function is given by

$$f(x,y) = -\tan^{-1}\left(\frac{x}{y}\right)$$



On the loop γ; $f(\gamma(\theta)) = -\tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$

θ is discontinuous at $(1,0)$

$$= -\tan^{-1}\left(\tan\left(\frac{\pi}{2}-\theta\right)\right)$$

$$= \theta - \frac{\pi}{2},$$

where θ is not a continuous function on $\mathbb{R}^2\setminus\{0\}$.
Hence f cannot be a potential function of F on $\mathbb{R}^2\setminus\{0\}$.

3) Since $\oint_C \vec{F}\cdot d\vec{\gamma} = 2\pi \neq 0$, $\nexists$ potential function f s.t. $F = \nabla f$ on $\mathbb{R}^2\setminus\{0\}$.

Curl and Green's Thm:

Green's thm: Let $U \subseteq \mathbb{R}^2$ be an open, connected and simply connected region.
Suppose a simple closed curve C enclosed a closed and bounded region $D \subseteq U$ moving anticlockwisely.
If $F(x,y) = (M(x,y), N(x,y))$, then

$$\iint_D \operatorname{curl}\vec{F}\, dA = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \oint_C M\,dx + N\,dy$$

$$= \oint_C \vec{F}\, d\vec{\gamma}.$$

Example: 1) $\oint_C xy^2\,dx + 2x^2y\,dy, \quad C: 4x^2+9y^2=36$

$$= \iint_D (4xy - 2xy)\,dx\,dy$$

Use the transformation $\begin{cases} x = 3r\cos\theta & r\in[0,1] \\ y = 2r\sin\theta, & \theta\in[0,2\pi] \end{cases}$

$$= \iint_D 2xy\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 2(3r\cos\theta)(2r\sin\theta)(6r)\,dr\,d\theta$$

$$= 0$$

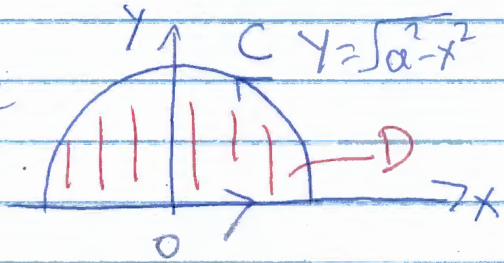
Exercises: 2) $\oint_C e^x\sin y\,dx + e^x\cos y\,dy$, where C is any simple closed curve with positive orientation.

Ans: $\oint_C e^x\sin y\,dx + e^x\cos y\,dy$

$$= \iint_D (e^x\cos y - e^x\cos y)\,dA$$

$$= 0$$

3) $\oint_C x^2y\,dx - y^2x\,dy$, where



Ans: $\oint_C x^2y\,dx - y^2x\,dy$

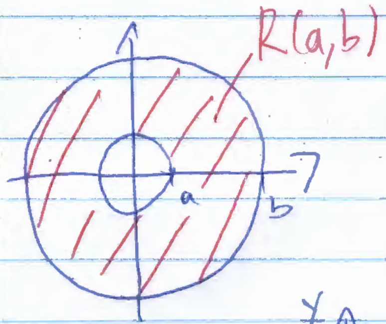
$$= \iint_D (-y^2 - x^2)\,dx\,dy$$

Polar coordinate

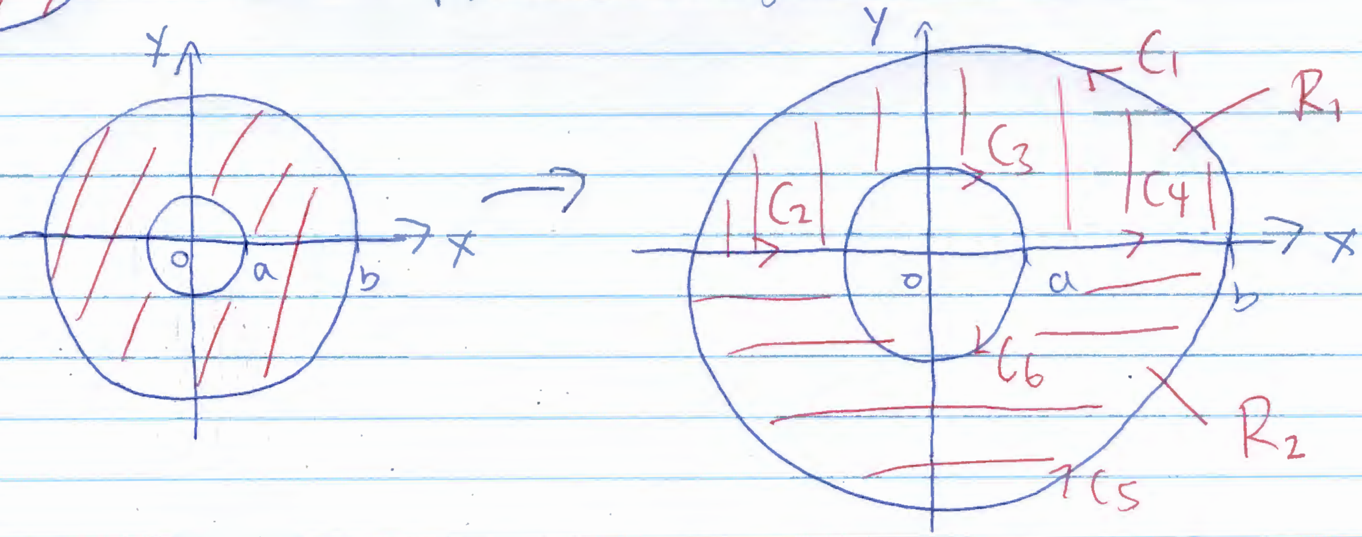
$$= \int_0^{\pi}\int_0^{a} (-r^2)(r)\,dr\,d\theta$$

$$= -\frac{a^4\pi}{4}$$

Example: 4) Let $R(a,b) = \{(x,y) \in \mathbb{R}^2 \mid a \leq \sqrt{x^2+y^2} \leq b\}$, where $0 < a < b < +\infty$.

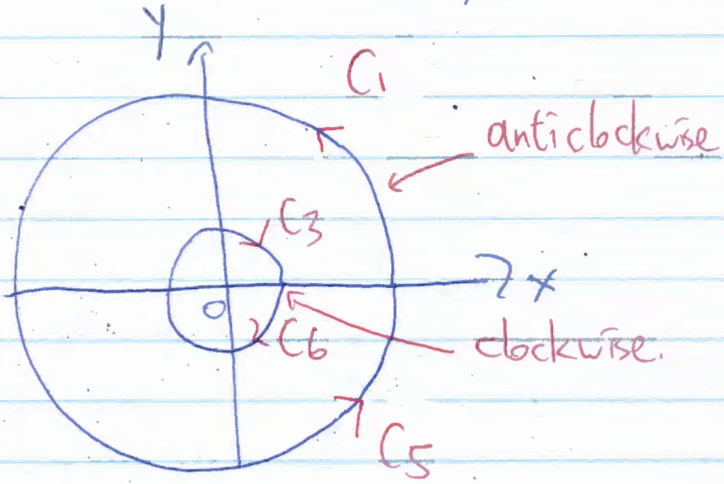


Green's thm does not apply.
Hence, we need to seperate $R(a,b)$ into two simply connected region R_1 and R_2 as follows:



where R_1 and R_2 are the regions bounded by $C_1 C_2 C_3 C_4$ and $C_5 (-C_4) C_6 (-C_2)$ respectively.

Then $$\iint_{R(a,b)} \text{curl}\,\vec{F}\,dA = \left(\iint_{R_1} + \iint_{R_2}\right) \text{curl}\,\vec{F}\,dA$$

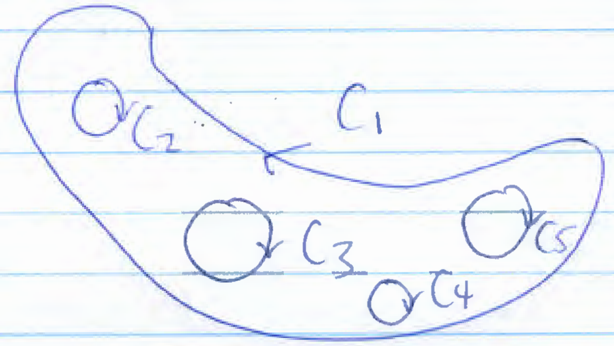


$$= \left(\oint_{C_1+C_2+C_3+C_4} + \oint_{C_5-C_4+C_6-C_2}\right) \vec{F}\,d\vec{r}$$

$$= \left(\oint_{C_1+C_5} + \int_{C_3+C_6}\right) \vec{F}\,d\vec{r}.$$

<u>Remark:</u> 1) This method can be generalized to other domain.

e.g.

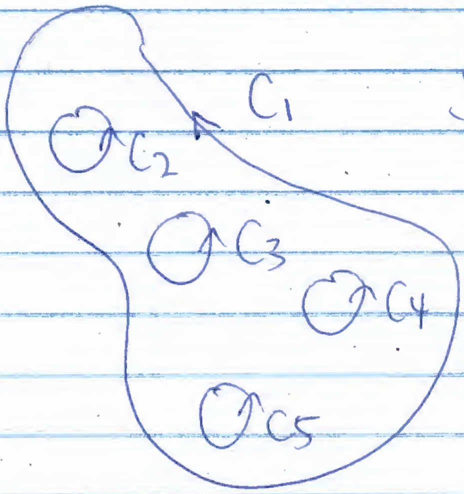


2) If the vector field $F(x,y) = (M(x,y), N(x,y))$ satisfies $\text{curl}\ \vec{F} = 0$, then we have

$$\oint_{C_1+C_5} \vec{F}\, d\vec{r} = \oint_{-C_3-C_6} \vec{F}\, d\vec{r}$$

both are in anticlockwise direction

Apply this to the domain below, by induction on the no. of 'holes', we have



$$\oint_{C_1} \vec{F}\, d\vec{r} = \sum_{i=2}^{5} \oint_{C_i} \vec{F}\, d\vec{r}.$$